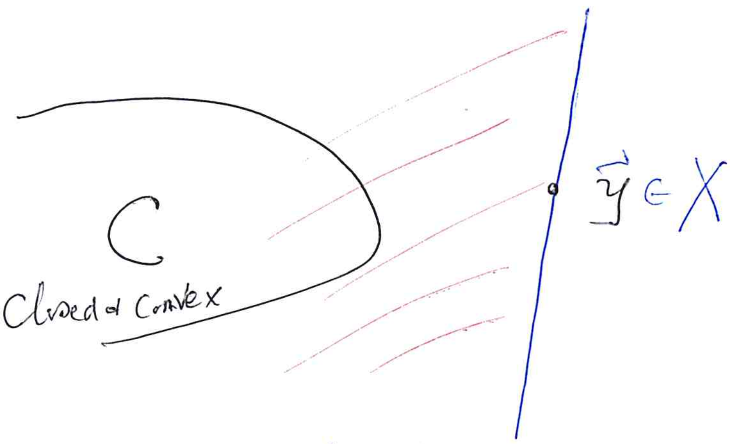
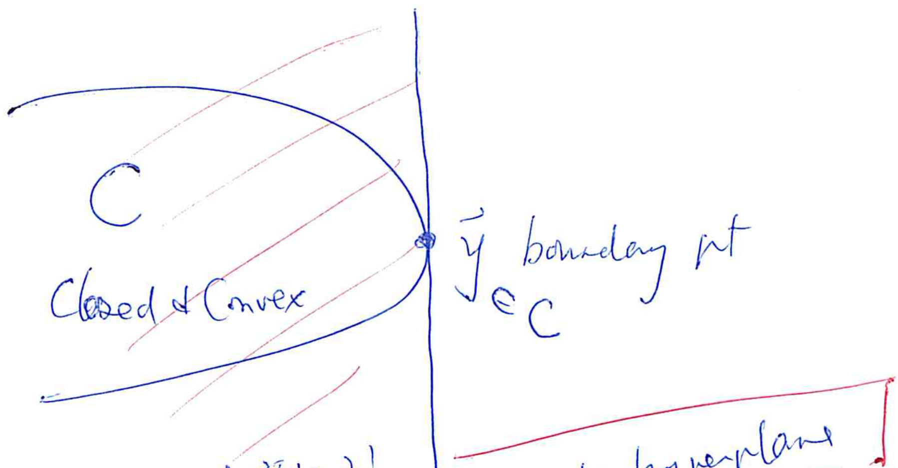
0

From last time

$X = \{\vec{x} \mid \vec{c}^T\vec{x} = z\}$



$C \subseteq X^+ = \{\vec{x} \mid \vec{c}^T\vec{x} > z\}$



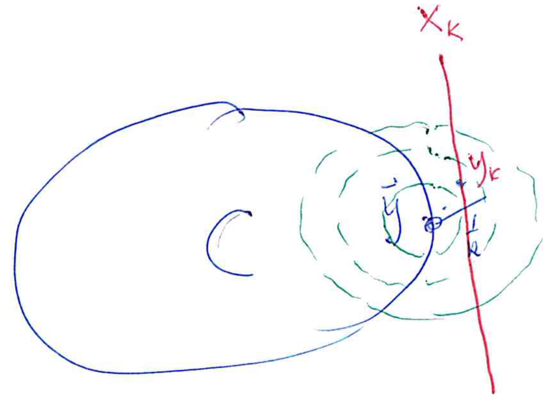
$C \subseteq X^+ = \{\vec{x} \mid \vec{c}^T\vec{x} \geq z\}$

$X = \{\vec{x} \mid \vec{c}^T\vec{x} = z\}$

$\vec{y} \in X$

Thm 1.4

Pf: $\vec{y} \in C$
boundary pt



$\forall k>0,\ B_{\frac{1}{k}}(\vec{y}) \ni \vec{y}_k$ & $\vec{y}_k \notin C$

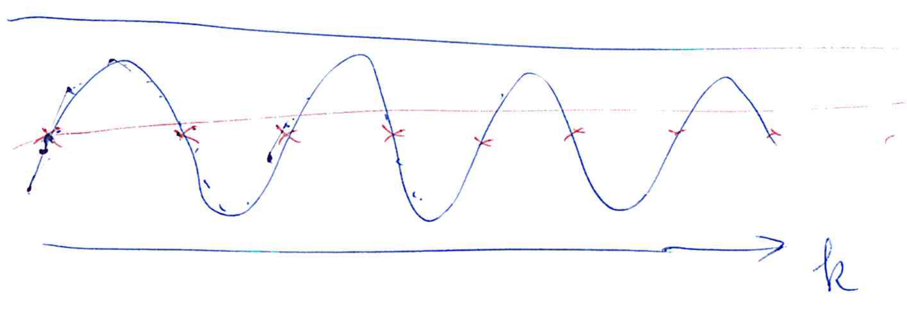
$k \to \infty \qquad \lim_{k\to\infty} \vec{y}_k = \vec{y}$ (1)

$\because \vec{y}_k \notin C$ $\boxed{\text{by Thm 1.3}}$ $\exists X_k = \{\vec{x} \mid \vec{a}_k^T \vec{x} = \vec{a}_k^T \vec{y}_k\} \ni \vec{y}_k$

& $C \subseteq X_k^+ = \{\vec{x} \mid \vec{a}_k^T \vec{x} \gneq \vec{a}_k^T \vec{y}_k\}$ (2)

w.l.o.g. $|\vec{a}_k| = 1$

$\{\vec{a}_k\}$ st $|\vec{a}_k| = 1 \overset{\text{subsequence}}{\Longrightarrow} \lim_{j\to\infty} \vec{a}_{k_j} = \vec{a}$ (Bounded Convergence Thm) (3)



$X = \{\vec{x} \mid \vec{a}^T \vec{x} = \vec{a}^T \vec{y}\} \ni \vec{y}$

claim it to be a supporting hyperplane

i.e. $C \subseteq X^+ = \{\vec{x} \mid \vec{a}^T \vec{x} \geq \vec{a}^T \vec{y}\}$ N.T.P.

$\textcircled{2}$

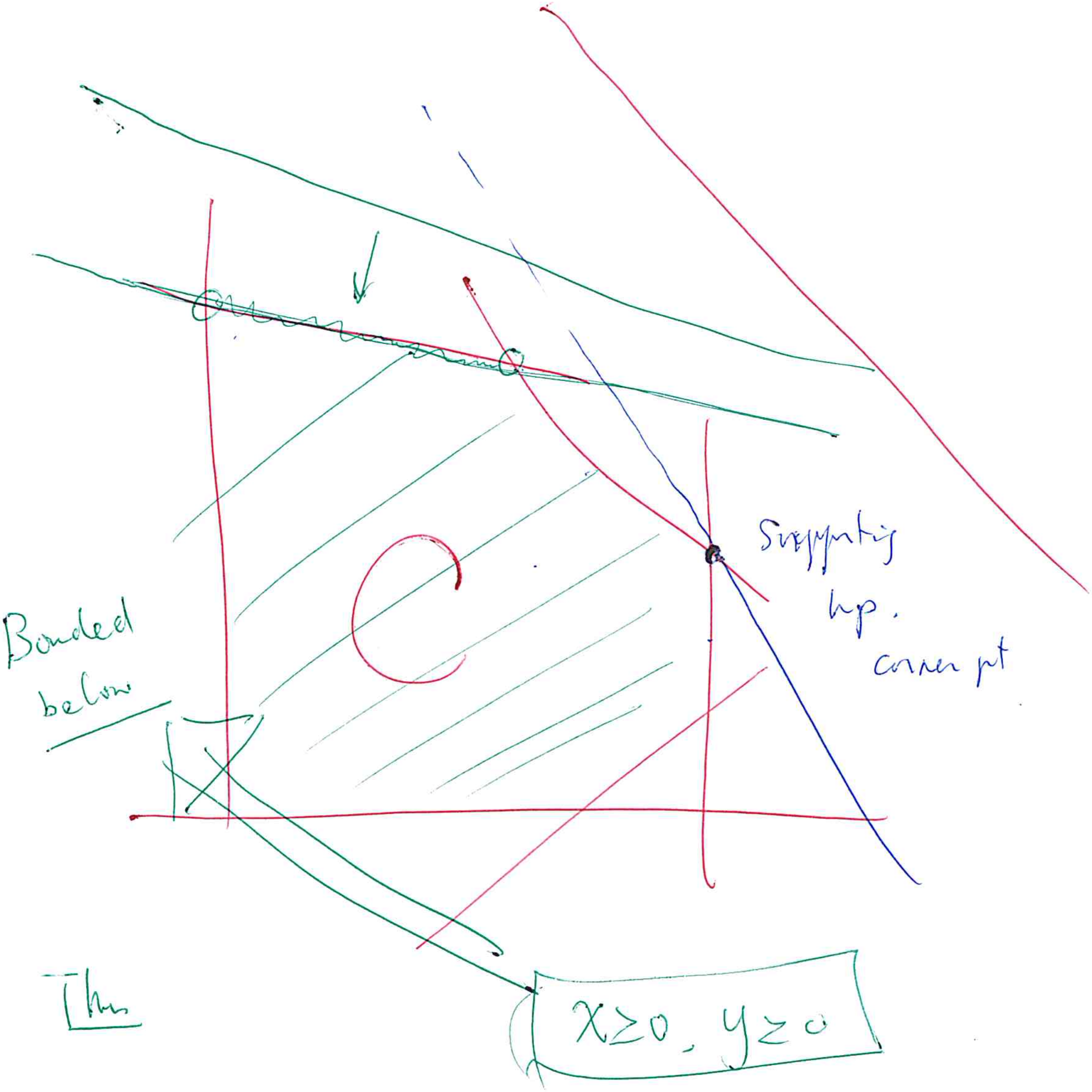
$\forall \vec{x} \in C$ n.t.p $\underline{\vec{a}^T \vec{x} \geq \vec{a}^T \vec{y}}$?!

$$\vec{a}^T \vec{x} \overset{3}{=} \lim_{j \to \infty} \vec{a}_{k_j}^T \cdot \vec{x} \qquad \vec{x} \in C, \vec{x} \in X_k^+$$

$$\overset{2}{\geq} \lim_{j \to \infty} \vec{a}_{k_j}^T \vec{y}_{k_j}$$

$$\overset{1}{=} \vec{a}^T \cdot \vec{y}$$

\#



Thm

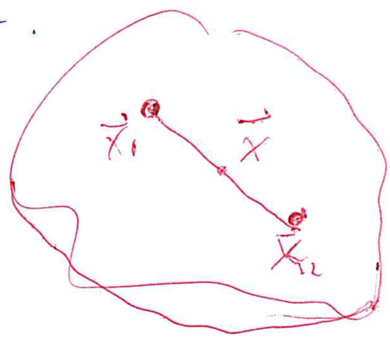
C closed & bdded from below
convex

Then every supporting hyperplane contains
an extreme pt of C.

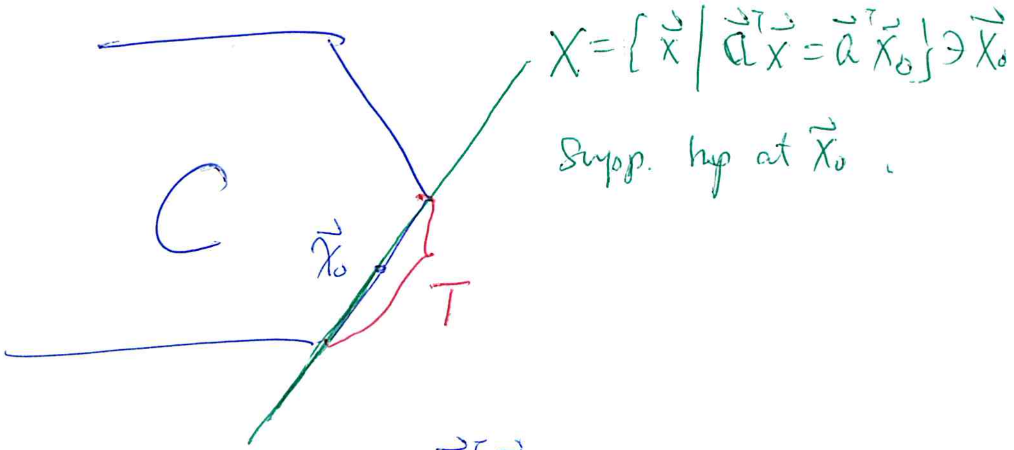
Def (ex. pt) ~~∃~~ $\vec{x}$ is ex. pt of C if

$\nexists$ $\vec{x}_1, \vec{x}_2 \in C$ s.t.

$\vec{x} = \lambda \vec{x}_1 + (1-\lambda)\vec{x}_2 \quad \lambda \in (0,1).$



pf:



$\vec{x}_0 \in C$ is bddary pt $\vec{a}^T\vec{x}_0 = z$

w.l.o.g. $C \subseteq X^+ = \{\vec{x} \mid \vec{a}^T\vec{x} \geq z\}$ (Thm. 1.4)

Def $T = X \cap C$

(i) $T \neq \phi$ $\vec{x}_0 \in C$, $\vec{x}_0 \in X \Rightarrow \vec{x}_0 \in T$ $T \neq \phi$. $\boxed{\text{done}}$

(ii) ex. pt of $T \Rightarrow$ ex pt of C

(iii) $\exists$ an ex pt of T.

---

pf of (ii) not an ex pt of $C \Rightarrow$ not an ex pt of T.

Let $\vec{t}$ not an ex pt of C.

Case 1: $\vec{t} \notin T$ then $\vec{t}$ cannot be an ex pt of T

Case 2 $\vec{t} \in T$; $\exists\ \vec{x}_1, \vec{x}_2 \in C$ s.t.

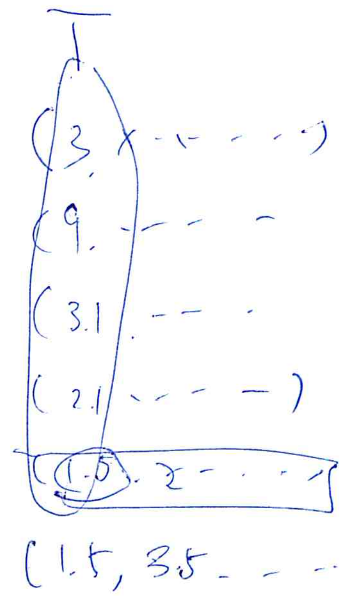
$$\vec{t} = \lambda \vec{x}_1 + (1-\lambda)\vec{x}_2 \qquad \lambda \in (0,1)$$

$$\left.\begin{aligned} \vec{a}^T\vec{t} &= \lambda \underbrace{\vec{a}^T\vec{x}_1}_{\geq z} + (1-\lambda)\underbrace{\vec{a}^T\vec{x}_2}_{\geq z} \\ \overset{\|}{z} &= \lambda \cdot z + (1-\lambda) \cdot z \end{aligned}\right\} \Rightarrow \begin{aligned} \vec{a}^T\vec{x}_1 = z &\Rightarrow \vec{x}_1 \in X \\ \vec{a}^T\vec{x}_2 = z &\Rightarrow \vec{x}_2 \in X \end{aligned}$$

$\therefore\ \vec{t} = \lambda \overset{\in T}{\vec{x}_1} + (1-\lambda)\overset{\in T}{\vec{x}_2} \Rightarrow \vec{t}$ is not ex pt of T

(iii) $\exists$ an ext pt of T

$\vec{t} \in T$ $\quad \vec{t} = (t_1, t_2, t_3, \ldots t_n)$



$$t^1 = \min_{\vec{t} \in T} t_1$$

t^1 is unique

$$t^2 = \min_{\substack{\vec{t} \in T \\ t_1 = t^1}} t_2$$

t^2 is unique

$\vdots$

t^j is unique $\longleftarrow$ an extreme pt of T.

at or before n step